Mathematics
Faculté des sciences d'Orsay

Sheet 11

Groups and geometry
M2 AAG 2025–2026

Curvature

Exercise 1 Conformal sectional curvature and Laplacian.
Let g be the standard Riemannian metric on $\mathbb{R}^2$ and $U \subset \mathbb{R}^2$ be an open subset. On U we consider $g' = \varphi g$ where $\varphi = U \to \mathbb{R}$ is smooth.

1. Under what condition on φ does g' define a Riemannian metric on U?

Let $\varphi = e^{2\sigma}$.

2. Let ∇ denote the Levi-Civita connection associated with the metric g and ∇' the connection associated with g'. Show that for all $X, Y \in \Gamma(TU)$,

$$\nabla'_X Y = \nabla_X Y + (X \cdot \sigma)Y + (Y \cdot \sigma)X - g(X,Y)\nabla^g \sigma$$

   where $\nabla^g \sigma$ is the vector field defined by $T_x\sigma(h) = g_x(\nabla^g \sigma(x), h)$ for all $h \in T_xU$.

3. Let $\partial_x = \frac{\partial}{\partial x}$ and $\partial_y = \frac{\partial}{\partial y}$ be the coordinate vector fields. Compute, in terms of σ, the Christoffel coefficients of the connection ∇'.
4. Give the sectional curvature of g' in terms of the Laplacian of σ, $\Delta\sigma$.
5. Prove (again ?) that the hyperbolic metric of the upper half-plane model of $\mathbb{H}^2_{\mathbb{R}}$ has curvature -1.

Exercise 2 Jacobi fields for surfaces and curvature.
Let (M, g) be a Riemannian manifold of dimension 2, x a point in M, and U a neighborhood of the origin in T_xM on which $\exp_x$ is a diffeomorphism onto its image. We consider the metric $h = \exp_x^* g$ on U. Using Gauss lemma, h can be expressed in polar coordinates (r, θ) on $U \setminus \{0\}$ as $h = dr^2 + f^2(r,\theta)d\theta^2$.
Let u, v be an oriented orthonormal basis for h_0. We assume that the polar coordinates have been chosen such that the angle θ associated with u is zero.

1. Let $c : r \mapsto ru$. Let J be the Jacobi field along c with initial conditions $J(0) = 0$ and $\frac{\nabla}{dr}J(0) = v$. Prove $J(r) = f(r,0)V(r)$ where $r \mapsto V(r)$ is the parallel transport of v along c.
2. Deduce that the sectional curvature of h at $c(r)$ is $-\frac{\partial_r^2 f(r,0)}{f(r,0)}$.
3. Compute (again ?) the sectional curvature of $\mathbb{E}^2$, $\mathbb{S}^2$, and (bonus) $\mathbb{H}^2$.

Exercise 3 Sectional curvature and length of small circles.
Let (M, g) be a Riemannian manifold and x a point in M. Let P be a 2-plane in T_xM and (u, v) an orthonormal basis of P. We set $H(r,\theta) = \exp_x\Big(r\cos(\theta)u + r\sin(\theta)v\Big)$ for $0 < r < \mathrm{inj}_x$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. For such r, we denote the curve $\theta \mapsto H(r,\theta)$ by C_r.

1. Show that, for every θ_0, $J_{\theta_0} : r \mapsto \frac{d}{d\theta}\Big|_{\theta=\theta_0} H(r,\theta)$ is a Jacobi field.
2. Show that $\|J_\theta(r)\| = r - \frac{K(P)}{6}r^3 + o(r^3)$.
3. Prove that $L(C_r) = 2\pi r\left(1 - \frac{K(P)}{6}r^2 + o(r^2)\right)$.

4. Compute again the curvature of $\mathbb{S}^2$ by computing directly $L(C_r)$.

Exercise 4 Theorema Egregium.
Let S be a (smooth) surface in $\mathbb{R}^3$ and $N : S \to \mathbb{S}^2$ be a (smooth) unit normal vector field for S. Recall that T_pN is self-adjoint and the product of its eigenvalues is called the Gaussian curvature K_G of S. Moreover, the second fundamental form $\mathrm{I\!I}$ is the quadratic form defined by $\mathrm{I\!I}_p(u,v) = -\langle T_pN \cdot u, v\rangle$ for all $p \in S$.
On S we consider the metric induced by the Riemannian metric on $\mathbb{R}^3$, ∇ the associated Levi-Civita connection and R its curvature.
Let $\nabla^{\mathbb{R}^3}$ be the Levi-Civita connection on $\mathbb{R}^3$ associated to the standard metric.

1. Let $X, Y \in \Gamma(TS)$ and extend arbitrarily X and Y in a neighborhood of S. Prove that

$$\nabla_X^{\mathbb{R}^3} Y = \nabla_X Y + \mathrm{I\!I}(X,Y)N.$$

2. Prove that, for all $X, Y, Z, W \in \Gamma(TS)$

$$R(X,Y,Z,W) = \mathrm{I\!I}(X,W)\,\mathrm{I\!I}(Y,Z) - \mathrm{I\!I}(X,Z)\,\mathrm{I\!I}(Y,W)$$

3. Prove that the sectional curvature is the Gaussian curvature (and therefore the Gaussian curvature of a surface is invariant under local isometry)