Sheet 10

Exercise 1 Lie groups and Riemannian geometry.
Let G be a (real) Lie group with Lie algebra $\mathfrak{g}$.

1. Prove there is a unique connection ∇ on TG such that, for any left-invariant vector fields X and Y on G, $\nabla_X Y = \frac{1}{2}[X, Y]$.
2. Prove that ∇ is torsion free.
3. (Preview) Compute $R(X, Y)Z$ for left-invariant vector fields X, Y and Z.
4. Prove that, for every X in $\mathfrak{g}$ and every $g \in G$, $t \mapsto g \exp_G(tX)$ is a geodesic for ∇. Prove that all geodesics have this shape.
5. In this question, we assume G is equipped with a Riemannian metric h that is invariant under all left and right translations (we say h is bi-invariant). We denote by h_1 the corresponding inner product on $\mathfrak{g}$.
   (a) Prove that, for every $g \in G$, h_1 is $\mathrm{Ad}(g)$-invariant.
   (b) Prove that, for every $X \in \mathfrak{g}$, $\mathrm{ad}(X)$ is antisymmetric on $\mathfrak{g}$ with respect to h_1.
   (c) Using the method used in class to prove uniqueness of the Levi-Civita connection, compute that Levi-Civita connection acting on left-invariant vector fields. Describe Riemannian geodesics.
   (d) (Preview) For every X and Y in $\mathfrak{g}$ with $\|X\| = \|Y\| = 1$ and $X \perp Y$, compute the sectional curvature of $\mathrm{Span}(X, Y)$.
6. We now assume G is compact.
   (a) Using a right-invariant volume form on G, prove that $\mathfrak{g}$ admits an inner product invariant under the adjoint action of G on $\mathfrak{g}$.
   (b) Prove that G admit a bi-invariant metric.
   (c) Prove $\exp_G$ is surjective.

Exercise 2 Convexity for small-radius balls.
Let (M, g) be a Riemannian manifold of dimension n and $m \in M$. Let $U \subset T_m M$ be a neighborhood of the origin such that $\exp_m : U \to \exp_m(U)$ is a diffeomorphism.
On U, we consider the Riemannian metric $h = \exp_m^* g$ and the associated distance d. We also consider the dot product h_0 at 0 and the associated norm $||u|| = h_0(u, u)^{\frac{1}{2}}$. This norm can be extended to $T_x U$ for all $x \in U$. Let $(e_1, \ldots, e_n)$ be an orthonormal basis for h_0. This basis gives us a basis of $T_x U$ for all $x \in U$.

1. For $x \in U$, the Christoffel tensor defines a bilinear map Γ_x on $T_x U$ with values in $T_x U$

$$\Gamma_x(u, v) = \sum_{k=1}^{n} \sum_{i,j=1}^{n} \Gamma_{i,j}^k(x) u_i v_j e_k.$$

   Show that there exists $\epsilon_0 \in (0, 1)$ such that if x belongs to B_{ε_0}, the ball centered at 0 with radius ϵ_0 for $\|\cdot\|$, we have $\|\Gamma_x(u, v)\| \leq \delta \|u\| \cdot \|v\|$ for some $0 \leq \delta < 1$.

2. Let $c : [0, a] \to U$ be a geodesic contained in B_{ϵ_0}. Assume that for some $0 < \epsilon < \epsilon_0$, and $t_0 \in (0, a)$, we have $c(t_0) \in S_\epsilon = \partial \overline{B}_\epsilon$, and $c'(t_0) \neq 0$ is tangent to S_ϵ. Prove that there exists $\lambda > 0$ such that for t sufficiently close to t_0, we have:

$$d(0, c(t))^2 \geq d(0, c(t_0))^2 + \lambda(t - t_0)^2.$$

Hint: Consider the function $F(t) = \frac{\|c(t)\|^2}{2}$ *and show that* $F''(t_0) > 0$.

3. Deduce that if $r > 0$ is small enough, the ball $\overline{B}_r$ is convex in the sense that if x and y are in $\overline{B}_r$, there exists a geodesic, unique up to affine reparameterization and minimizing, connecting x and y, which is entirely contained in $\overline{B}_r$.

Preview

Let g be the standard Riemannian metric on $\mathbb{R}^2$ and $U \subset \mathbb{R}^2$ be an open subset. On U we consider $g' = \varphi g$ where $\varphi = U \to \mathbb{R}$ is smooth.

1. Under what condition on φ does g' define a Riemannian metric on U?

Let $\varphi = e^{2\sigma}$.

2. Let ∇ denote the Levi-Civita connection associated with the metric g and ∇' the connection associated with g'. Show that for all $X, Y \in \Gamma(TU)$,

$$\nabla'_X Y = \nabla_X Y + (X \cdot \sigma)Y + (Y \cdot \sigma)X - g(X, Y)\nabla^g \sigma\,.$$

Where $\nabla^g \sigma$ is the vector field defined by $T\sigma_x(h) = g_x(\nabla^g \sigma(x), h)$ for all $h \in T_x U$.

3. Let $\partial_x = \frac{\partial}{\partial x}$ and $\partial_y = \frac{\partial}{\partial y}$ be the coordinate vector fields. Compute, in terms of σ, the Christoffel coefficients of the connection ∇'

4. Give the sectional curvature of g' in terms of the Laplacian of σ, $\Delta\sigma$.

5. Prove (again ?) that the hyperbolic metric of the upper half-plane model of $\mathbb{H}^2_{\mathbb{R}}$ has curvature -1.