Mathematics
Faculté des sciences d'Orsay

Sheet 10

Groups and geometry
M2 AAG 2025–2026

Exercise 1

Exercise 2 Convexity for small-radius balls.
Let (M,g) be a Riemannian manifold of dimension n and $m \in M$. Let $U \subset T_mM$ be a neighborhood of the origin such that $\exp_m : U \to \exp_m(U)$ is a diffeomorphism.
On U, we consider the Riemannian metric $h = \exp_m^* g$ and the associated distance d. We also consider the dot product h_0 at 0 and the associated norm $||u|| = h_0(u,u)^{\frac{1}{2}}$. This norm can be extended to T_xU for all $x \in U$. Let $(e_1, \dots, e_n)$ be an orthonormal basis for h_0. This basis gives us a basis of T_xU for all $x \in U$.

1. For $x \in U$, the Christoffel tensor defines a bilinear map Γ_x on T_xU with values in T_xU

$$\Gamma_x(u,v) = \sum_{k=1}^{n} \sum_{i,j=1}^{n} \Gamma_{i,j}^k(x) u_i v_j e_k.$$

   Show that there exists $\epsilon_0 \in (0,1)$ such that if x belongs to B_{ε_0}, the ball centered at 0 with radius ϵ_0 for $\|\cdot\|$, we have $\|\Gamma_x(u,v)\| \leq \delta \|u\| \cdot \|v\|$ for some $0 \leq \delta < 1$.

2. Let $c : [0,a] \to U$ be a geodesic contained in B_{ϵ_0}. Assume that for some $0 < \epsilon < \epsilon_0$, and $t_0 \in (0,a)$, we have $c(t_0) \in S_\epsilon = \partial \overline{B}_\epsilon$, and $c'(t_0) \neq 0$ is tangent to S_ϵ. Prove that there exists $\lambda > 0$ such that for t sufficiently close to t_0, we have:

$$d(0,c(t))^2 \geq d(0,c(t_0))^2 + \lambda (t-t_0)^2.$$

   Hint: Consider the function $F(t) = \frac{\|c(t)\|^2}{2}$ *and show that* $F''(t_0) > 0$.

3. Deduce that if $r > 0$ is small enough, the ball $\overline{B}_r$ is convex in the sense that if x and y are in $\overline{B}_r$, there exists a geodesic, unique up to affine reparameterization and minimizing, connecting x and y, which is entirely contained in $\overline{B}_r$.

Solution.

1. As $h = \exp_m^* g$, we have $\Gamma_{i,j}^k(0) = 0$ for all $1 \leq i,j,k \leq n$. Fix $\eta > 0$. As the $\Gamma_{i,j}^k$ are continuous, there exists ϵ_0 such that for all $x \in \overline{B}_{\epsilon_0}$ we have $\left|\Gamma_{i,j}^k(x)\right| \leq \eta$ for all $1 \leq i,j,k \leq n$. Then, for all $u,v \in T_xU$ we have

$$\|\Gamma_x(u,v)\| \leq \sum_{i,j,k=1}^{n} \left|\Gamma_{i,j}^k(x)\right| |u_i||v_j| \leq n^3 \eta \|u\| \|v\|.$$

   For $\eta < 1/n^3$, we obtain $\|\Gamma_x(u,v)\| \leq \delta \|u\|\|v\|$ with $\delta = n^3\eta < 1$.

2. Let $F : [0,a] \to \mathbb{R}$ be defined by $F(t) = \|c(t)\|^2/2$. The geodesic c is smooth and so is F. For all $t \in (0,a)$, we have

$$F(t) = \frac{h_0(c(t),c(t))}{2}$$
$$F'(t) = h_0(c'(t),c(t))$$
$$F''(t) = h_0(c''(t),c(t)) + h_0(c'(t),c'(t))$$

At $t = t_0$, $c'(t)$ is tangent to $S_\epsilon = \{u \in U, h_0(u,u) = \epsilon\}$. Therefore $h_0(c'(t_0), c(t_0)) = 0$. Moreover, as c is a geodesic, we have

$$c''(t_0) = -\sum_{i,j,k=1}^{n} \Gamma_{i,j}^k(c(t_0))c_i'(t_0)c_j'(t_0)e_k = -\Gamma_{c(t_0)}(c'(t_0), c'(t_0)).$$

Thus

$$h_0(c''(t_0), c(t_0)) = h_0\left(-\Gamma_{c(t_0)}(c'(t_0), c'(t_0)), c(t_0)\right).$$

By Cauchy-Schwarz inequality, the estimate in the previous question and the inequality $\epsilon_0 \leq 1$, we obtain

$$\begin{aligned}|h_0(c''(t_0), c(t_0))| &\leq \left\|\Gamma_{c(t_0)}(c'(t_0), c'(t_0))\right\| \|c(t_0)\| \\ &\leq \delta \left\|c'(t_0)\right\|^2.\end{aligned}$$

Thus

$$F''(t_0) \geq \left\|c'(t_0)\right\|^2 (1-\delta) > 0.$$

Using Taylor expansion, we obtain

$$F(t) = F(t_0) + \frac{F''(t_0)}{2}(t-t_0)^2 + o((t-t_0)^2)$$

and therefore

$$F(t) \geq F(t_0) + \lambda(t-t_0)^2$$

for t close enough to t_0 and $\lambda = \frac{F''(t_0)}{4}$. Now, as we use the exponential map, we have $d(0,x) = \|x\|$ for all $x \in U$ and therefore

$$d(0,c(t))^2 \geq d(0,c(t_0))^2 + \lambda(t-t_0)^2$$

for all t close enough to t_0.

3. Choose $r < \epsilon_0/3$ small enough so that for all $x, y \in \overline{B}_r$, there exists a unique minimizing geodesic (up to affine reparameterization) connecting x and y in U. Let $x, y \in \overline{B}_r$. We have $d(x,y) \leq d(0,x) + d(0,y) \leq 2r$ and therefore the minimizing geodesic $c : [0,a] \to U$ connecting x and y is contained in $\overline{B}_{3r} \subset B_{\epsilon_0}$.
Let t_0 be such that $\|c(t)\|$ is maximal at t_0 (such a t_0 exists by compactness). If $t_0 \in (0,a)$, then $c'(t_0)$ is tangent to $S_{\|c(t_0)\|}$ and we can use the previous question. Therefore, for t close to t_0, we have

$$d(0,c(t))^2 \geq d(0,c(t_0))^2 + \lambda(t-t_0)^2$$

and therefore

$$d(0,c(t)) > d(0,c(t_0))$$

for t close to t_0 and $t \neq t_0$. This is a contradiction. Therefore $t_0 \in \{0,a\}$ and $c([0,a]) \subset \overline{B}_r$.