Sheet 9

Parallel transport

Exercise 1 Parallel transport on a cone and on a sphere.

1. Describe the parallel transport along a curve c in $\mathbb{R}^2$ (for the standard connection).
2. In $\mathbb{R}^3$ with Cartesian coordinates (x, y, z), consider a half-line D from the origin in the xz plane at angle α with Oz and the revolution cone C of axis Oz that it generates. Let $c : [a, b] \to C$ be a horizontal circle and X be a parallel vector field along c. Compute the angle between $X(a)$ and $X(b)$.
   Hint. Unfold C to obtain a flat surface in $\mathbb{R}^2$.
3. (Bonus) Same question for a parallel vector field along a small circle on the sphere S^2.

Geodesics

Exercise 2 Geodesics.

1. Show that geodesics on $\mathbb{R}^n$ are straight lines parametrized at constant velocity.
2. Show that the geodesics on a Riemannian n-manifold $M \subset \mathbb{R}^{n+p}$ are the curves with normal acceleration vector field (i.e. the field of acceleration vectors is everywhere normal to M).

Exercise 3 Geodesics on $\mathbb{S}^2$ and the hyperbolic hyperboloid.
Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere with the metric induced by the Euclidean metric on $\mathbb{R}^3$.

1. Let $N = (0, 0, 1)$ be the north pole of $\mathbb{S}^2$. Let $u \in T_N\mathbb{S}^2$ with $u \neq 0$. Let γ be the geodesic starting at N with initial velocity u. Let P be the plane generated by $(0, 0, 1)$ and u (seen as a vector in $\mathbb{R}^3$).
   (a) Prove that γ is contained in P.
   (b) Prove that γ travel along the great circle $P \cap \mathbb{S}^2$ at constant speed.
   (c) Describe the geodesics of $\mathbb{S}^2$
2. Use the same method to prove that the geodesics of the hyperbolic hyperboloid H are the intersection of H with 2-planes through the origin with constant speed parametrization (see the next section for more informations on H).
3. Using the previous question, describe the geodesics of the hyperbolic disk.
   Hint. Use the invariance by rotation to study $H \cap P$ where P is given by $y = 0$ or $by - z = 0$ with $|b| > 1$ and use the inverse map of the stereographic projection.

The hyperbolic hyperboloid from Exercise 2 - Sheet 7

In $\mathbb{R}^3$, we consider H the upper sheet ($z > 0$) of the two-sheeted hyperboloid $z^2 - x^2 - y^2 = 1$. On H, we consider the metric g induced by the Minkowski metric $dx^2 + dy^2 - dz^2$ in $\mathbb{R}^3$. This is a Riemannian metric on H. Let $S = (0, 0, -1)$ and $B(O, 1)$ be the unit ball in $\mathbb{R}^2$. One can define $\varphi : H \to B(O, 1)$, an hyperbolic stereographic projection from S . We have

$$\varphi(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

and

$$\varphi^{-1}(u, v) = \left(\frac{2u}{1-u^2-v^2}, \frac{2v}{1-u^2-v^2}, \frac{1+u^2+v^2}{1-u^2-v^2}\right).$$

Moreover

$$\varphi_* g = \frac{4}{(1-u^2-v^2)^2} g_0$$

where g_0 is the standard metric on $\mathbb{R}^2$. Therefore $(B(O, 1), \varphi_* g)$ is the hyperbolic disk.
Recall that $O(2, 1)$ is the subgroup of matrices preserving the Minkowski quadratic form

$$O(2,1) = \left\{ M \in M_3(\mathbb{R}), {}^t M \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} M = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Let $O_+(2, 1)$ be the subgroup of $O(2, 1)$ preserving H. Then $O_+(2, 1)$ acts isometrically on H, acts transitively on 2-planes through the origin intersecting H and acts transitively on orthonormal basis on H.

Preview

Let G be a (real) Lie group with Lie algebra $\mathfrak{g}$.

1. Prove there is a unique connection ∇ on TG such that, for all left-invariant vector fields X and Y on G, $\nabla_X Y = \frac{1}{2}[X, Y]$.
2. Prove that ∇ is torsion free.
3. Prove that, for every X in $\mathfrak{g}$ and every $g \in G$, $t \mapsto g \exp_G(tX)$ is a geodesic for ∇. Prove that all geodesics have this shape.

References

Exercise 1. S. Gallot, D. Hulin and J. Lafontaine. *Riemannian Geometry.* 2.76
Exercise 3. John M. Lee. *Riemannian Manifolds.* Chapter 5. Section *Geodesics of the Model Spaces*