Mathematics
Faculté des sciences d'Orsay

Sheet 9

Groups and geometry
M2 AAG 2025–2026

Geodesics

Exercise 1

Exercise 2

Exercise 3 Geodesics on $\mathbb{S}^2$ and the hyperbolic hyperboloid.
Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere with the metric induced by the Euclidean metric on $\mathbb{R}^3$.

1. Let $N = (0,0,1)$ be the north pole of $\mathbb{S}^2$. Let $u \in T_N\mathbb{S}^2$ with $u \neq 0$. Let γ be the geodesic starting at N with initial velocity u. Let P be the plane generated by $(0,0,1)$ and u (seen as a vector in $\mathbb{R}^3$).
   (a) Prove that γ is contained in P.
   (b) Prove that γ travel along the great circle $P \cap \mathbb{S}^2$ at constant speed.
   (c) Describe the geodesics of $\mathbb{S}^2$
2. Use the same method to prove that the geodesics of the hyperbolic hyperboloid H are the intersection of H with 2-planes through the origin with constant speed parametrization (see the next section for more informations on H).
3. Using the previous question, describe the geodesics of the hyperbolic disk.
   Hint. Use the invariance by rotation to study $H \cap P$ where P is given by $y = 0$ or $by - z = 0$ with $|b| > 1$ and use the inverse map of the stereographic projection.

Solution.

1. (a) Let $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$ be the symmetry with P as plane of symmetry. Then σ is an isometry so $\gamma_\sigma = \sigma(\gamma)$ is a geodesic. As $\gamma_\sigma(0) = N$ and $\gamma'_\sigma(0) = \sigma(u) = u$ and as a geodesic is uniquely determined by initial value and initial velocity we have $\gamma = \gamma_\sigma$. Therefore γ is contained in P.
   (b) Since geodesics have constant speed, γ travel along the great circle $P \cap \mathbb{S}^2$ at constant speed.
   (c) As $O(3)$ acts isometrically (and therefore sends geodesics to geodesics) and transitively on orthonormal basis of $T\mathbb{S}^2$, the geodesics of $\mathbb{S}^2$ are the great circles (ie the intersections of $\mathbb{S}^2$ with 2-planes through the origin) with constant speed parametrization
2. Let $N = (0,0,1)$. Note that $T_N H \simeq \mathbb{R}^2 \times \{0\}$. Fix $u \in T_N H$ with $u \neq 0$ and let γ be the geodesic with initial value N and initial velocity u. Let P be the plane generated by $(0,0,1)$ and u. We want to prove that γ is contained in $H \cap P$. Let σ be symmetry with P as plane of symmetry. As σ is the identity on Oz and is an isometry in Oxy then $\sigma \in O(2,1)$. Moreover, $\sigma(N) = N$ and therefore $\sigma \in O_+(2,1)$. Thus σ is an isometry of H and $\sigma(\gamma)$ is a geodesic. We conclude using the uniqueness of geodesics as before. Therefore γ travel along $H \cap P$ at constant speed. As $O_+(2,1)$ acts isometrically and transitively on orthonormal basis, we deduce that geodesics of H are the $H \cap \sigma(P)$ for $\sigma \in O_+(2,1)$. As $O_+(2,1)$ acts transitively on 2-planes through the origin intersecting H, we obtained the desired result.

3. Fix P such that $P \cap H \neq \emptyset$. Then $P \cap Oxy$ is 1 dimensional and, as rotations with axis Oz are in $O_+(2,1)$, we may assume $Ox = P \cap Oxy$. Therefore P can be describe by the equation $y = 0$ or by the equation $by - z = 0$ for some $b \in \mathbb{R}$. As $P \cap H \neq \emptyset$, we have $b^2 > 1$, ie $|b| > 1$.

— Case $y = 0$. Then P is a vertical plane and its stereographic projection on $B(0,1)$ is a straight line through the origin.

— Case $by - z = 0$. Let $(u, v) = \varphi(x, y, z)$ with $(x, y, z) \in H \cap P$. Then $z = by$ and, using, the inverse map of the stereographic projection, we obtain

$$2bv = 1 + u^2 + v^2$$

with is equivalent to

$$u^2 + (v - b)^2 = b^2 - 1$$

so our geodesic is contained in the circle $\mathcal{C}$ of center $B = (0, b)$ and radius $\sqrt{b^2 - 1}$. Conversely, any point in $\mathcal{C} \cap B(O, 1)$ is sent to a point in $P \cap H$ by φ^{-1}. Moreover, let A be in $\mathcal{C} \cap S(O, 1)$. Then Pythagora's theorem tells us that OAB is a right angled triangle at A. Therefore our geodesic is a circular arc orthogonal to $S(O, 1)$. All the circles with center on Oy and orthogonal to $S(O, 1)$ are of the previous form. Therefore, all the circular arcs with center of Oy and orthogonal to $S(O, 1)$ are geodesic arcs given by some P as above.

Thus the geodesics of the hyperbolic disk are the straight lines through the origin and circular arc orthogonal to the boundary.

The hyperbolic hyperboloid from Exercise 2 - Sheet 7

In $\mathbb{R}^3$, we consider H the upper sheet ($z > 0$) of the two-sheeted hyperboloid $z^2 - x^2 - y^2 = 1$. On H, we consider the metric g induced by the Minkowski metric $dx^2 + dy^2 - dz^2$ in $\mathbb{R}^3$. This is a Riemannian metric on H. Let $S = (0, 0, -1)$ and $B(O, 1)$ be the unit ball in $\mathbb{R}^2$. One can define $\varphi : H \to B(O, 1)$, an hyperbolic stereographic projection from S . We have

$$\varphi(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

and

$$\varphi^{-1}(u, v) = \left(\frac{2u}{1 - u^2 - v^2}, \frac{2v}{1 - u^2 - v^2}, \frac{1 + u^2 + v^2}{1 - u^2 - v^2}\right).$$

Moreover

$$\varphi_* g = \frac{4}{(1 - u^2 - v^2)^2} g_0$$

where g_0 is the standard metric on $\mathbb{R}^2$. Therefore $(B(O, 1), \varphi_* g)$ is the hyperbolic disk.
Recall that $O(2, 1)$ is the subgroup of matrices preserving the Minkowski quadratic form

$$O(2,1) = \left\{ M \in M_3(\mathbb{R}), {}^t M \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} M = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Let $O_+(2, 1)$ be the subgroup of $O(2, 1)$ preserving H. Then $O_+(2, 1)$ acts isometrically on H, acts transitively on 2-planes through the origin intersecting H and acts transitively on orthonormal basis on H.