Sheet 8

Connections

Exercise 1 Dual connection. Let ξ be a vector bundle over a manifold M, endowed with a connection ∇. Show that the formula:

$$(\nabla^*_X \varphi)(\sigma) = X \cdot (\varphi(\sigma)) - \varphi(\nabla_X \sigma)$$

for $X \in \Gamma(TM)$, $\varphi \in \Gamma(\xi^*)$, and $\sigma \in \Gamma(\xi)$ defines a connection ∇^* on ξ^*.

Exercise 2 Canonical connection of a Riemannian submanifold. Using the coordinates $F : (\theta, \varphi) \mapsto (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$ for the sphere $\mathbb{S}^2$ (endowed with the Levi-Civita connection ∇ associated to the Riemannian metric inherited from the Euclidean metric in $\mathbb{R}^3$), compute

$$\nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\theta}, \qquad \nabla_{\frac{\partial}{\partial\varphi}} \frac{\partial}{\partial\theta}, \qquad \nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\varphi}, \qquad \nabla_{\frac{\partial}{\partial\varphi}} \frac{\partial}{\partial\varphi}.$$

Exercise 3 Connections on a trivial bundle. Let $U \subset \mathbb{R}^m$ be an open set and $p : U \times \mathbb{R}^n \to U$ be the associated trivial fiber bundle with fiber $\mathbb{R}^n$. Describe all the (Koszul) connections on $U \times \mathbb{R}^n$.

Back to vector bundles

Exercise 4 $L(E, F)$. Let M be a manifold ans E and F be two vector bundles over M. Describe transitions maps for local trivializations for the vector bundle $L(E, F)$.

Exercise 5 Pull back of fiber bundles.

1. Let E be the tangent bundle to $\mathbb{S}^2$ and $f : \mathbb{S}^1 \to \mathbb{S}^2$ be the map $\theta \mapsto (\cos(\theta), \sin(\theta), 0)$. Show that f^*E is a trivial bundle.
2. (Bonus) Let $E = \{(e^{i\theta}, w) \in \mathbb{S}^1 \times \mathbb{C}, w \in \mathbb{R}e^{i\theta/2}\}$. Let $p : E \to \mathbb{S}^1$ be the projection to the first factor.
   (a) Prove E is well-defined.
   (b) Identify E.
   (c) Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be defined by $f(z) = z^2$. Describe f^*E.

Preview

1. Describe the parallel transport along a curve in $\mathbb{R}^2$ (for the standard flat connection).
2. In $\mathbb{R}^3$ with Cartesian coordinates (x, y, z), consider a half-line from the origin in the xz plane at angle $\alpha \in [0, \pi/2]$ with Oz and the revolution cone C of axis Oz that it generates. Let $c : [a, b] \to C$ be a horizontal circle and X be a parallel vector field along c. Compute the angle between $X(a)$ and $X(b)$.
   Hint. Unfold C to obtain a flat surface in $\mathbb{R}^2$.

References

Exercise 2. S. Gallot, D. Hulin and J. Lafontaine. *Riemannian Geometry*. Exercises 2.57 (Be careful - typo $\nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\varphi} = \cos(\theta)\sin(\theta)\frac{\partial}{\partial\theta} = \frac{1}{2}\sin(2\theta)\frac{\partial}{\partial\theta}$)