Sheet 7

Riemannian metrics

Exercise 1 The sphere. Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere. We consider the metric on $\mathbb{S}^2$ inherited form the Euclidean metric on $\mathbb{R}^3$.

1. Give a formula for the stereographic projection from the north pole $\varphi : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$ and for its inverse map.
2. Let $\overline{g}$ be the metric on $\mathbb{R}^2$ induced from the metric on $\mathbb{S}^2$ by the stereographic projection.
   (a) Compute $\overline{g}$ using the definition of the pull-back metric.
   (b) Compute $\overline{g}$ using more geometric arguments.
3. Show that $O(3)$ acts isometrically on $\mathbb{S}^2$ and acts transitively on orthonormal basis on $\mathbb{S}^2$.

Exercise 2 The hyperbolic hyperboloid. In $\mathbb{R}^3$, we consider H the upper sheet ($z > 0$) of the two-sheeted hyperboloid $z^2 - x^2 - y^2 = 1$. On H, we consider the metric induced by the Minkowski metric $dx^2 + dy^2 - dz^2$ in $\mathbb{R}^3$. Note that the Minkowski metric is a Lorentz metric not a Riemannian metric but we will prove that h is a Riemannian metric.

1. Let $S = (0, 0, -1)$. Let $B(0, 1)$ be the unit ball in $\mathbb{R}^2$. Define $\varphi : H \to B(0, 1)$, an hyperbolic stereographic projection from S . Give a formula for φ and its inverse map.
2. Let $\overline{g}$ be the metric on $B(0, 1)$ induced from the metric on H by the hyperbolic stereographic projection. Compute $\overline{g}$.
3. Let $O_+(2, 1)$ be the subgroup of $O(2, 1)$ preserving H. Show that $O_+(2, 1)$ acts isometrically on H and acts transitively on orthonormal basis on H.

Exercise 3 The cone. (Bonus.) Let $C = \{(x, y, z) \in \mathbb{R}^3 | z^2 = x^2 + y^2, z > 0\}$ be an open cone. We consider on C the metric g inherited from the Euclidean metric on $\mathbb{R}^3$.

1. Write the standard metric on $\mathbb{R}^2$ in polar coordinates.
2. Let $p : C \to \mathbb{R}^2 \setminus \{(0, 0)\}$ be the restriction to C of the projection $(x, y, z) \mapsto (x, y)$. Determine the induced metric $\overline{g} = p_* g$ on $\mathbb{R}^2 \setminus \{(0, 0)\}$ in polar coordinates by calculation and by a geometric argument.

Vector bundles

Exercise 4 $L(E, F)$. Let M be a manifold ans E and F be two vector bundles over M. Describe transitions maps for local trivializations for the vector bundle $L(E, F)$.

Exercise 5 Triviality of vector bundles. (Bonus.)

1. Let γ_n be the tautological bundle over the projective space $\mathbb{RP}^n = P(\mathbb{R}^{n+1})$ (with $n \geq 1$). Recall that its total space is $\{(x, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} | v \in x\}$.

(a) Show that if s is a nowhere-vanishing continuous section of γ_n, then there exists a continuous map $t : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^*$ such that $s(\mathbb{R}v) = t(v)v$.

(b) Show that the vector bundle γ_n is not trivial.

2. Show that the tangent bundle of S^3 is trivial.
   Hint. Use Exercise 1 in Sheet 1 (on Lie groups).

3. Let ξ be a real vector bundle of rank 1 over M. Show that the bundle $\mathrm{End}(\xi)$ of endomorphisms of ξ is always trivial.

Preview of Sheet 8

1. Let ξ be a vector bundle over a manifold M, endowed with a connection ∇. Show that the formula:
$$(\nabla^*_X \varphi)(\sigma) = X \cdot (\varphi(\sigma)) - \varphi(\nabla_X \sigma)$$
   for $X \in \Gamma(TM)$, $\varphi \in \Gamma(\xi^*)$, and $\sigma \in \Gamma(\xi)$ defines a connection ∇^* on ξ^*.

2. Using the coordinates $(\theta, \varphi) \mapsto (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$ for the sphere $\mathbb{S}^2$, compute $\nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\theta}$.

References

Exercices 1 and 2. John M. Lee. *Riemannian Manifolds.* Chapter 3. Section *The Model Spaces of Riemannian Geometry*