Mathematics | Sheet 6 | Groups and geometry
Faculté des sciences d'Orsay | | M2 AAG 2025–2026

Sheet 6

Curvature of curves and surfaces

Exercise 1 Curvature of plane curves. Let $\gamma : I \to \mathbb{R}^2$ be a (smooth) curve parametrized with respect to arc length (i.e. $T(s) = \gamma'(t)$ is a unit tangent vector). Let $N : I \to \mathbb{R}^2$ be a unit normal vector field along γ such that $(T(s), N(s))$ is a direct basis for all $s \in I$. Recall that the *signed curvature* κ is defined by

$$\gamma''(s) = \kappa(s)N(s).$$

1. Why is $\gamma''(s)$ collinear to $N(s)$?
2. Compute the curvature of a circle of radius R.
3. Compute $N'(s)$. Does the obtained formula remain correct if we change the parameterization of γ ?
4. Let γ be a curve of constant curvature. Prove that the image of γ is contained in a line or in a circle.
5. (Bonus) Let $s \mapsto (x(s), y(s))$ be a (smooth) immersed parametric curve. Compute its curvature.

Exercise 2 Gaussian curvatures of surfaces in $\mathbb{R}^3$. Let S be a (smooth) surface in $\mathbb{R}^3$ and $N : S \to \mathbb{S}^2$ be a (smooth) unit normal vector field for S.

1. Let $p \in S$. Explain why T_pN can be seen as an endomorphism of T_pS.
2. Let $p \in S$. Show that T_pN is a self-adjoint linear map.

As T_pN is self-adjoint, it is diagonalizable in an orthonormal basis. Let $-k_1(p)$ and $-k_2(p)$ be its eigenvalues and $(e_1(p), e_2(p))$ an orthonormal basis of eigenvectors. Then k_1 and k_2 are called the *principal curvatures* and k_1k_2 is called the *Gaussian curvature* of S. Let $\mathrm{I\!I}$ be the quadratic form, called the *second fundamental form*, defined by

$$\mathrm{I\!I}_p(u, v) = -\langle T_pN \cdot u, v\rangle$$

for all $p \in S$. Notice that if we change N to $-N$, we obtain opposite principal curvatures but the same Gaussian curvature.

3. Let $p \in S$. Write $\mathrm{I\!I}_p$ in the basis $(e_1(p), e_2(p))$.
4. Compute $\mathrm{I\!I}$, the principal curvatures and the Gaussian curvature of
   (a) the sphere of radius R;
   (b) the cylinder of axis (Oz) and radius R;
   (c) the hyperbolic paraboloid $z = y^2 - x^2$ at $(0, 0, 0)$.
5. Let P be a affine plane such that $p \in P$ and $N(p) \in \overrightarrow{P}$.
   (a) Show that $S \cap P$ is a smooth curve in a neighborhood of p.

   Let γ_P be this curve. We choose an orientation for γ_P compatible with N.

(b) Compute the curvature of γ_P at p using II_p.

(c) Prove that $k_1(p)$ and $k_2(p)$ are the maximum and minimum of the curvature of the curves γ_P for all P as above.

6. Let $\gamma : I \to S$ be a (smooth) curve. We say γ is a *geodesic* for S if $\gamma''(s)$ is collinear to $N(\gamma(s))$ for all $s \in I$.

   (a) Show that the parallels of the sphere $\mathbb{S}^2$ (ie the great circles intersecting the poles) parametrized at constant speed 1 are geodesics of $\mathbb{S}^2$.

   (b) Let C be the cylinder of axis (Oz) and radius 1. Let $p = (1, 0, 0) \in C$. Give a geodesic with initial point p and initial velocity $v_1 = (0, 1, 0)$. Same question with $v_2 = (0, 0, 1)$ and (Bonus) $v_3 = (0, u_0, v_0)$.

7. Assume that S is connected and for all $p \in S$, we have $k_1(p) = k_2(p) = \kappa$. Show that S is contained in a plane or in a sphere.

8. (Bonus) Assume that S is connected and for all $p \in S$, $k_1(p) = k_2(p)$. Show that the principal curvatures are constant.

9. (Bonus) Let $p \in S$ and P be an affine plane such that $p \in P$. Let $\gamma : I \to S \cap P$ be an immersed curve such that $\gamma(0) = p$. Compare the curvature of γ at 0 and $\mathrm{II}(\gamma'(0))$.

10. (Bonus - This question may involve more computations than the previous ones.)

    (a) Compute the principal curvatures and the Gaussian curvature of a surface of revolution obtained by rotating $u \mapsto (r(u), 0, z(u))$ around the z-axis (where $r > 0$ and $u \mapsto (r(u), z(u))$ is an immersion).

    (b) Compute the principal curvatures of a torus of revolution given by the circle of radius r_2 and center $(r_1, 0)$ (where $0 < r_2 < r_1$).

    (c) Give a geometric interpretation of k_1 and k_2 (for the torus or in the general case of a surface of revolution)

    (d) Compute the Gaussian curvature of the surface of revolution given by the tractrix $u \mapsto (1/\cosh(u), u - \tanh(u))$.

References

For Exercise 2 - Questions 1 to 9. Manfredo P. do Carmo. *Diffrential Geometry of Curves and Surfaces.* Chapter 3.

For surfaces of revolution: *Pierre Pansu's webpage Cours de Géométrie Différentielle*
`https://www.imo.universite-paris-saclay.fr/~pierre.pansu/web_dea/resume_dea_04.html`