CONTACT HOMOLOGY OF CONTACT ANOSOV FLOWS

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Abstract. In this short note we prove that if a contact structure admits a contact Anosov flow with orientable foliations then the number of Reeb periodic orbits of any non-degenerate contact form grows exponentially with the period. This result applies to Foulon and Hasselblatt's construction. It hinges on a computation of contact homology.

After completing this note, I learned that Macarini and Paternain [7] *proved a similar result for fillable Anosov contact structures using symplectic homology.*

The goal of this short note is to study the consequences on Reeb dynamics of the existence of contact Anosov flows. It is motivated by the recent construction of such flows by Foulon and Hasselblatt [4] using surgery along a Legendrian knot and generalizing Handel-Thurston [6] and Goodman [5] works. It is a conjecture of Colin and Honda [2, Conjecture 2.10] that the number of Reeb periodic orbits of universally tight contact structures on hyperbolic manifolds grows exponentially with the period. Foulon and Hasselblatt's constructions contain examples of contact Anosov flows on such manifolds and this note proves the conjecture for these contact structures.

We first recall some basic definitions. A 1-form α on a 3-manifold M is called a *contact form* if $\alpha \wedge \mathrm{d}\alpha$ is a volume form. The associated plane field $\xi = \ker(\alpha)$ is a cooriented *contact structure*. A curve tangent to ξ is called *Legendrian*. The *Reeb vector field* associated to a contact form α is the vector field R_α such that $\iota_{R_\alpha}\alpha = 1$ and $\iota_{R_\alpha}\mathrm{d}\alpha = 0$. Its flow is called the *Reeb flow*. The Reeb flow is *non-degenerate* if all its periodic orbits are non-degenerate. A flow is said to be *contact* if it is the flow of some Reeb vector field. The flow ϕ^t generated by X on a Riemannian manifold M is an *Anosov flow* if the tangent bundle TM has a ϕ^t invariant splitting into $E^\phi \oplus E^s \oplus E^u$ and there exist $\lambda, C > 0$ such that E^ϕ is generated by X and for all $t > 0$, $\|D\phi^t_{|E^s}\| \leq Ce^{-\lambda t}$ and $\|D\phi^{-t}_{|E^u}\| \leq Ce^{-\lambda t}$. The 1-dimensional foliations E^s and E^u are called the *strong stable and unstable foliations*. Note that if ϕ^t is a contact Anosov flow associated to the contact form α, then $\ker(\alpha) = E^s \oplus E^u$. For more information on the properties of Anosov flows, on can refer to [9].

The *growth rate of contact homology* is the main tool to control Reeb periodic orbits of all the contact forms associated to a given contact structure (see for instance [2, 8]). *Contact homology* is an invariant of the contact structure computed through a Reeb vector field and introduced in the vein of Morse and Floer homology by Eliashberg, Givental and Hofer in 2000 [3]. The associated complex is generated by Reeb periodic orbits and the differential "counts" holomorphic cylinders. The *growth rate of contact homology* [1] consists in a filtration by the action (i.e. the period). It "describes" the asymptotic behavior with respect to T of the number of Reeb periodic orbits with period smaller than T that contribute to contact homology. The growth rate of contact homology lower bounds the growth rate of Reeb period orbits for any contact form (for which contact homology is well defined) and thus has deep dynamic consequences. For a more detailed presentation of contact homology and its growth rate, one can refer to [8].

Theorem 1. *Let (M, α) be a closed contact 3-manifold. If the Reeb flow is Anosov with orientable strong stable and unstable foliations, then the cylindrical contact homology is well-defined and generated by the R_α-periodic orbits. Additionally, the growth rate of contact homology is exponential. The stable and unstable foliations in Foulon-Hasselblatt's construction* [4] *are orientable.*

Corollary 2. *Under the hypothesis of Theorem 1, for any non-degenerate contact form α' without contractible periodic orbit, the number of $R_{\alpha'}$-periodic orbits with period smaller than T grows exponentially with T.*

Remark 3. Though commonly accepted, existence and invariance of contact homology remain unproved in general. See [8, Hypothesis H] for more details. Corollary 2 hinges only on proved results. If one assumes general existence and invariance of contact homology, Corollary 2 generalizes to non-degenerate contact forms α' (without the assumption on contractible periodic orbits).

The end of the note is devoted to the proof of the main theorem and the review of some helpful properties of contact homology and Foulon-Hasselblatt's construction. We use only the simplest version of contact homology: cylindrical contact homology over $\mathbb{Q}$. This contact homology is well-defined for primitive class of periodic orbits provided there is no contractible Reeb periodic orbit. This condition is always satisfied by Anosov flows in dimension 3.

Let ϕ^t denote the Reeb flow and consider a non degenerate periodic orbit γ with period T and a point p on γ. The map $\mathrm{d}\phi^T(p) : (\xi_p, \mathrm{d}\alpha) \to (\xi_p, \mathrm{d}\alpha)$ is a symplectomorpism and γ is called *even* if $\mathrm{d}\phi^T(p)$ has two real positive eigenvalues and *odd* otherwise. The proof hinges on the following fundamental property of contact homology. The differential of an odd (resp. even) orbit contains only even (resp. odd) orbits.

We prove that, under the assumptions of Theorem 1, the Reeb periodic orbits are even and hyperbolic. By definition of stable and unstable foliations, $D\phi^T_{|\xi}(p)$ has real eigenvalues μ and $\frac{1}{\mu}$ and the associated eigenspaces are E^u and E^s. Additionally, the inequality

$$\frac{1}{|\mu|^k} = \|D\phi^{kT}_{|E^s}\| \leq Ce^{-\lambda kT}$$

implies $|\mu| > 1$. As the strong stable foliation is orientable the eigenvalues are positive. Thus γ is even and hyperbolic and the differential is trivial. The exponential growth of contact homology derives from the exponential growth of periodic orbits of Anosov flows. This concludes the proof of the first part of Theorem 1.

Before turning to the second part of the theorem, we recall the mains steps of Foulon-Hasselblatt's construction. Contact Anosov flows are obtained by surgery along a Legendrian curve γ in a unitary tangent bundle over a hyperbolic surface. There exist coodinates $(s,w,t) \in S^1 \times (-\epsilon,\epsilon) \times (-\eta,\eta)$ in a neighborhood of γ such that $\alpha = \mathrm{d}t + w\mathrm{d}s$. The surgery is performed along $S^1 \times (-\epsilon,\epsilon) \times \{0\}$ and described by the gluing map

$$\begin{array}{rccc} F: & S^1 \times (-\epsilon,\epsilon) & \longrightarrow & S^1 \times (-\epsilon,\epsilon) \\ & (s,w) & \longmapsto & (s+f(w),w). \end{array}$$

where a lift of f is a cut-off function between 0 and $-2q\pi$. The new contact form is given by $\alpha_h = \alpha + \mathrm{d}h$ for $t \geq 0$ and $\alpha_h = \alpha - \mathrm{d}h$ for $t < 0$ where h is an explicit smooth function such that $F^*(\alpha - \mathrm{d}h) = \alpha + \mathrm{d}h$. The Reeb vector field is collinear to $\frac{\partial}{\partial t}$. The Reeb flow is Anosov and the proof of this property (due to Barbot) relies on the construction of Lyapunov-Lorentz metrics (see [4, Appendix A]). In the previous coordinates, the Lyapunov-Lorentz metrics are

$$Q^{\pm} = \pm \mathrm{d}w\mathrm{d}s - c\mathrm{d}t^2, \qquad \text{if } t < 0,$$

$$Q^{\pm} = \pm\left(\mathrm{d}w\mathrm{d}s + \beta(t)f'(w)\mathrm{d}w^2\right) - c\mathrm{d}t^2, \qquad \text{if } t > 0,$$

where β is a cut-off function.

The strong stable foliation is contained in the positive cone of Q^- and the strong unstable foliation in the positive cone of Q^+. Therefore, stable foliation is orientable if and only if the positive cone of Q^- is orientable. The stable and unstable foliations of the unitary tangent bundle over a hyperbolic surface are orientable. Additionally, $Q^-\left(\frac{\partial}{\partial s},\frac{\partial}{\partial s}\right) = 0$ and $F^*\frac{\partial}{\partial s} = \frac{\partial}{\partial s}$. Thus the surgery preserves the orientation of Q^- and Q^- is orientable. This concludes the proof of Theorem 1

Corollary 2 derives from the invariance of contact homology as the growth rate of contact homology lower bounds the growth rate of the Reeb periodic orbits. Remark 3 hinges on the invariance of the linearized contact homology. See Theorem 1.2 and its proof in [8] for more details.

Acknowledgements

I would like to thank Patrick Foulon for his explanations on the construction of contact Anosov flows.

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